Quantum spectrum and statistic entropy of black hole

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Abtract

Taking the horizon surface of the black hole as a compact membrane and solving the oscillation equation of this membrane by Klein-Gordon equation, we derive the frequencies of oscillation modes of the horizon surface, which are proportional to the radiation temperature of the black hole. However, the frequencies of oscillation modes are not equidistant. Using the distribution of obtained frequencies of oscillation mode we compute the statistic entropy of the black hole and obtain that the statistic entropy of the black hole is proportional to the area of the horizon. Therefore, it is proven that the quantum statistic entropy of the black hole is consistent with Bekenstein-Hawking entropy.

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The necessity for a quantum theory of gravity was already recognized in the 1930s. However, despite the flurry of activity on this subject we still lack a complete theory of quantum gravity. It is believed that black holes may play a major role in our attempts to shed some light on the nature of a quantum theory of gravity (such as the role played by atoms in the early development of quantum mechanics) [1].

Quantization of the black hole horizon area is a long-standing problem. Since it has been proposed in 1974 by Bekenstein, this problem is entirely unsolved [2]. It involves the quantum origin of the black hole entropy. After Hod derived the interval between quantum spectrums of Schwarzschild black hole using Bohr correspondence principle, considerable progress in this field

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had been done in the past few years [3-6]. However, the interval of spectrums given by Refs. [1, 7] are equidistant. For complex space-time, we only obtain the numerical solution to frequency.

A basis for the Hilbert space of loop gravity is given by spin networks. These are graphs whose edges are labeled by representations of the gauge group of the theory. In the case of gravity this group is taken to be SU(2) and the representations are thus labeled by positive half-integers $j = 0, 1/2, 1, 3/2, \cdots$. If a surface is intersected by an edge of such a spin network carrying the label j the surface acquires the area [8,9]

$$A(j) = 8\pi l_p^2 \gamma \sqrt{j(j+1)}, \tag{1}$$

where l_p is the Planck length and γ is the so-called Immirzi parameter. Ref.[9] has given the value of parameter γ.

In this paper, taking the horizon surface of the black hole as a compact membrane and supposing that on the membrane the amplitude of wave is much smaller than the radius of horizon, we constitute the wave equation describing the propagation of surface wave on horizon surface. Comparing this equation with the angular equation of Klein-Gordon equation followed by the massless particles in the curved space-time, we can obtain the vibration frequency of the surface wave propagation on the black hole surface. This frequency is not equidistant and its quantum characteristic is similar to the quantum property of the black hole area given by Eq.(1).To simplify the discussion, we take $c = \hbar = l_p = K_B$.

The line element of the Schwarzschild black hole is

$$ds^2 = -A(r)dt^2 + A^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2}$$

where $A(r) = 1 - 2M/r$, M is the mass of the black hole.

In curved space-time, Klein-Gordon equation followed by massless particles is

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right) \Psi = 0. \tag{3}$$

Substituting the metric (2) into Eq.(3) and separating the variables based on the symmetry of the space-time, we have

$$\Psi(t, r, \theta, \varphi) = f(r)\chi(\theta)\Phi(\varphi)\exp(-iEt) = f(r)\chi(\theta)\exp(-i(Et - m\varphi)). \tag{4}$$

The radial and angular components satisfy the following equation [10, 11]

$$\frac{E^2}{A(r)}f(r)+\frac{1}{r^2}\frac{d}{dr}\left[r^2A(r)\frac{df(r)}{dr}\right]-\frac{l(l+1)}{r^2}f(r)=0, \tag{5}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\chi(\theta)}{d\theta}\right]+\left((l(l+1)-\frac{m^2}{\sin^2\theta}\right)\chi(\theta)=0, \tag{6}$$

where $l=0,1,2,3\cdots$, and $l\geq|m|$, E is the energy of the particle, m is a constant.

If we take the horizon surface of the black hole as a compact spherical membrane, we can discuss Eq.(3) and obtain the wave equation of the surface wave propagation on the spherical surface. To avoid divergent term in the equation, firstly, we calculate the wave equation of the surface wave propagation near horizon on $R=r_H+\varsigma$ the spherical surface (r_H is the location of the black hole horizon, ς is a positive small quantity). Secondly, letting $\varsigma\to 0$, we derive the wave equation of the surface wave propagation on the black hole horizon surface. When R is invariant, Eq.(3) can be reduced to

$$\frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi(\theta,\varphi,t)}{\partial\theta}\right)+\frac{1}{R^2\sin^2\theta}\frac{\partial^2\psi(\theta,\varphi,t)}{\partial\varphi^2}=\frac{1}{A(R)}\frac{\partial^2\psi(\theta,\varphi,t)}{\partial t^2}, \tag{7}$$

where the field function $\psi(\theta,\varphi,t)$ must be single-valued and continuous everywhere. So the field function satisfies the following conditions.

$$\chi(\theta)=\chi(\theta+2\pi),\quad \Phi(\varphi)=\Phi(\varphi+2\pi). \tag{8}$$

using Separation of Variables ,we obtain

$$\psi(\theta,\varphi,t)=\chi(\theta)\Phi(\varphi)T(t), \tag{9}$$

so

$$T(t)=B\sin(2\pi\nu t+\delta), \tag{10}$$

where δ is a phase constant, ν is a variation frequency. Eq.(7) can be rewritten as

$$\frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\chi(\theta)}{\partial\theta}\right)+\left(\frac{4\pi^2\nu^2}{A(R)}-\frac{m^2}{R^2\sin^2\theta}\right)\chi(\theta)=0. \tag{11}$$

at any point the field function satisfies the Eq.(6) on the background of the black hole, the field function satisfies Eq.(11). So When in Eqs.(6) and (11) we let $r = R$, they should be equivalence. Comparing Eq.(6) with Eq.(11), we obtain

$$l(l+1) = \frac{4\pi^2 \nu^2 R^2}{A(R)}, \tag{12}$$

Since equation (6) should be consistent with equation (11),we obtain a group of frequencies of oscillation modes, that is

$$\nu_l = \frac{A^{1/2}(R)}{2\pi R}\sqrt{l(l+1)}. \tag{13}$$

The frequency shift of the photon that is from the surface of the star got by the observer at rest at an infinite distance is as follows,

$$\nu = \nu^0 A^{1/2}(r), \tag{14}$$

where ν^0 is the natural vibration frequency on the surface of the star, ν is the natural frequency of this particle measured by the observer at rest at an infinite distance. Substituting (14) into (13) and letting $R \to r_H$, we derive that the natural frequency of the black hole horizon surface is

$$\nu_l^0 = \frac{1}{2\pi r_H}\sqrt{l(l+1)}. \tag{15}$$

Since $|m|$ can take value from 1 to l, and every m is double degenerate, if we add the mode $m = 0$, every oscillation mode has $(2l + 1)$ degenerate. In addition, the partial tone frequency is not the times of the lowest frequency $\nu_1^0 = \frac{\sqrt{2}}{2\pi r_H}$ corresponding $l = 1$. Mode frequency of $l = 0$ is zero. It represents the spherically symmetric mode under the case of non-vibration.

Introducing partition function Z

$$Z = \sum_l (2l+1) e^{-\frac{\beta}{2\pi r_H}\sqrt{l(l+1)}}, \tag{16}$$

where $\beta = 1/T_H$ is the inverse Hawking radiation temperature of the black hole. Taking l as continuous distribution and calculating (16), we have

$$Z \approx \int_0^\infty 2x e^{-\frac{\beta}{2\pi r_H}x} dx = \frac{2(2\pi r_H)^2}{\beta^2}. \tag{17}$$

The entropy of the black hole is

$$S = N\left(\ln Z - \beta\frac{\partial \ln Z}{\partial \beta}\right) = N\left(2 + \ln\frac{2(2\pi r_H)^2}{\beta^2}\right), \tag{18}$$

where N is the number of the particles. The internal energy of the black hole is

$$U = M = -N\frac{\partial \ln Z}{\partial \beta} = \frac{2N}{\beta}. \tag{19}$$

Substituting the radiation temperature of the black hole $T_H = 1/(4\pi r_H)$ and (19) into (18), we derive that the entropy of the black hole is

$$S = \frac{A}{4}(2 - \ln 2) = \frac{A}{4} \times 1.3, \tag{20}$$

where A is the horizon area of the black hole. As a result, we derive the relation between the quantum statistical entropy and the horizon area. The difference between the result of Eq.(20) and Bekenstein-Hawking entropy is caused by the fact that the integral replaces the sum in the calculation of the partition function. The accurate result should be derived by taking the sum of partition function.

According to Ref.[9], let the particles be at $l = 1$ quantum state. We have

$$Z = 3e^{-\frac{\beta}{2\pi r_H}\sqrt{2}}. \tag{21}$$

The entropy of the black hole is

$$S_{l=1} = \frac{A}{4}\frac{\ln 3}{\sqrt{2}}. \tag{22}$$

Because of $\frac{\ln 3}{\sqrt{2}} < 1$, $S_{l=1} < A/4$. That is the lower limit of the black hole entropy is $\frac{A}{4}\frac{\ln 3}{\sqrt{2}}$.

From (1), we have

$$\Delta A = A(j) = 8\pi\gamma\sqrt{j(j+1)}. \tag{23}$$

Since the area A and the M of a Schwarzschild black hole are related by

$$A = 16\pi M^2, \tag{24}$$

Comparing (23) with (24), we obtain

$$\Delta M = \frac{\gamma}{4M}\sqrt{j(j+1)}. \tag{25}$$

Comparing (25) with (15), we derive $\gamma = 1/\pi$. As a result, we derive that the natural vibration frequency of the black hole horizon surface (15) is equivalence with the spectrum of the black hole area given by (1). However, in our calculation there is not any uncertain factor.

Studying the radiation spectrum of the black hole is a very interesting subject. However, until recently, the energy of the radiation particles is derived by numerical calculations. For rotating black hole, the calculation is more complex. In this paper, taking the horizon surface of the black hole as a compact membrane and solving the oscillation equation of this membrane, we derive the frequencies of oscillation modes of the horizon surface. Then using the derived frequencies, we compute the black hole entropy. There is only small difference between our result and Bekenstein-Hawking entropy. This departure is caused mostly by the fact that the sum in Eq.(16) is turned to the integral in Eq.(17). We obtain the statistical entropy of the black hole under the case that there is not any assumption. Our result is very close to Bekenstein-Hawking entropy which shows that the quantum statistical entropy is consistent with the thermodynamic entropy. In our paper, we derive the analytical solution of the vibration frequency and in addition there is not any uncertain parameter. We provide a new method for studying the quantum characteristic of rotating black hole as well as more complex black hole.

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